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Graphs with Four Independent Crossings Are Five Colorable

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GRAPHS WITH FOUR INDEPENDENT CROSSINGS ARE FIVE COLORABLE

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Abstract. Albertson conjectured that if a graph can be drawn in the plane in such a way that any two crossings are independent, then the graph can be 5-colored. He proved it for up to three independent crossings. We prove this for four crossings by showing that any such graph has an independent set of size 4 with one vertex in each crossing, and give an example to show that this method fails for five independent crossings.

1. Introduction

A *proper (vertex) coloring* of a graph is a mapping from the vertex set of of the graph to a set of colors in such a way that if two vertices share an edge then they are not mapped to the same color. A *5-coloring (n-coloring)* of a graph is a proper coloring of a graph where the set of colors has 5 (n) elements. Given a graph G with a proper coloring, the m-n Kempe chain (where m and n are colors) through a vertex v is the maximum connected subgraph of G containing v that is only colored with the colors m and n. For an m-n Kempe chain, *switching the colors* means that we recolor the vertices colored m by the color n, and recolor the vertices originally colored n by the color m. Note that switching the colors on a Kempe chain produces another proper coloring of the graph.

A *drawing* of a graph on the plane is a mapping of the graph into $\mathbf{R}^2$ where vertices are mapped to points and edges are mapped to curves beginning and ending at the points corresponding to the two vertices in the edge, with certain standard restrictions.[1] A graph is called *planar* if it can be drawn in the plane with no crossings. For more basic graph theory terms and definitions see [2] or any other basic graph theory book. In this paper we assume that adjacent edges do not cross[2], so each crossing *involves* exactly four vertices. Two crossings are said to be *independent* if no vertex is involved in both crossings.

[1] An edge only intersects vertices at its ends. Edges only intersect at a crossing or at a shared endpoint. Three edges may not cross at a single point.

[2] This is true for a drawing that minimizes the number of crossings.

A recent paper of Albertson [1] considers the effect of crossings on chromatic number of a graph drawn in the plane. In particular, he investigates graphs with small numbers of crossings and independent crossings. He conjectured that if a graph can be drawn in the plane in such a way that any two crossings are independent, then the graph can be 5-colored. He proved the theorem for up to three crossings, by showing a stronger statement: in such a graph there is an independent set (with no edges) of three vertices, one from each crossing. Removing these vertices makes the graph planar, so it can thus be 4-colored by the four color theorem. Here we take the same approach as Albertson, trying to find an independent set of vertices, one from each crossing. We show that for four crossings we can always find such a set. For five we construct a counterexample where you cannot find such an independent set, although the graph is still 5-colorable.

Recently we learned that that Paul Wenger independently obtained the same results as we present in this paper (by a different proof), and that Dan Kral and Ladislav Stacho have recently proved Albertson's full conjecture from [1].

2. Results

Theorem 2.1. *If a graph G is drawn with exactly four crossings, which are pairwise independent, then G has an independent set with one vertex involved in each crossing.*

Proof. Consider the subgraph of G induced by the 16 vertices involved in the crossings. An independent set in this subgraph is also an independent set in G, so it suffices to prove the theorem for this subgraph. Thus, we may assume that G has exactly 16 vertices. It suffices to show that G has a 5-coloring: since there are 16 vertices one color must be used four times, once in each crossing.

Consider the 16-vertex graph G is drawn in the plane with four independent crossings. Without loss of generality each crossing is part of a copy of K_4 (see [1] for details). Using Euler's formula, we find that there are at most 22 edges not contained in a K_4, so there must exist a vertex v with degree at most 5. $G - v$ has only 3 independent crossings, so by Albertson's work [1] it can be 5-colored, with the color set $\{1, 2, 3, 4, 5\}$. If v is only adjacent to four colors then color it with the fifth color and G is five colored. Thus we assume that v is adjacent to five colors, and without loss of generality the neighborhood of v looks like this (showing all edges in the crossing that contains v):

Now consider the 5-2, 5-1, 4-3, and 4-1 Kempe chains that are adjacent to v. If any of them do not contain two neighbors of v, then switch the colors on that Kempe chain; then v is only adjacent to four colors so G can

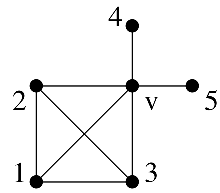

FIGURE 1. The neighbors of v and the crossing containing v.

be 5-colored. Otherwise, each of these Kempe chains must contain a path between two neighbors of v. Take such a path in the 5-2 Kempe chain and extend it through v to get a cycle drawn as a Jordan curve: call it $C_{5,2}$. Similarly define $C_{5,1}$ $C_{4,3}$ $C_{4,1}$. Now locally near v, $C_{4,3}$ starts on one side of $C_{5,2}$ and ends on another, so this forces a crossing with vertices labeled (4,5,3,2) in cyclic order. By a similar argument we must also have a (4,5,1,2) and a (4,5,3,1) crossing. Since there are only four crossings these three along with the one that involves v make up all the crossings.

By the Jordan curve theorem, $C_{5,2}$ divides the plane into two regions. We call the region containing the neighbor of v colored 4 to be the 5-2-*interior*, and the other region to be the 5-2-*exterior*. Now consider the set of non-neighbors of v that are colored 1 and 3: there are exactly two colored 1 and two labeled 3, as every vertex of G is involved in one of the four crossings. We have the following cases for which of these vertices are on the 5-2-interior and 5-2-exterior of $C_{5,2}$.

Case 0	Int. none	Ext. 1,1,3,3
Case 1	Int. 1	Ext. 1,3,3
Case 2	Int. 3	Ext. 1,1,3
Case 3	Int. 1,1	Ext. 3,3
Case 4	Int. 3,3	Ext. 1,1
Case 5	Int. 1,3,3	Ext. 1
Case 6	Int. 1,1,3	Ext. 3
Case 7	Int. 1,3	Ext. 1,3
Case 8	Int. 1,1,3,3	Ext. none

Case 0:

This cannot occur, since the neighbor of v colored 4 is only in one crossing, and both $C_{4,1}$ and $C_{4,3}$ must cross $C_{5,2}$.

Cases 1–6:

These either have two 1's on one side and one 3 on the other side or vice versa. These three vertices together with v form an independent set of size four, since two vertices colored the same do not share an edge, and the vertices colored 1 and 3 on opposite sides of $C_{5,2}$ do not share an edge, and because none of them are adjacent to v.

Case 7:

Consider where the (4,5,3,1) crossing is with respect to the $C_{5,2}$. Note that its vertex colored 5 may be in $C_{5,2}$, but its vertices colored 1 and 3 must be on the same side of $C_{5,2}$. If $(4, 5, 3, 1)$ is (mostly) on the 5-2-exterior, then consider the vertex colored 4 adjacent to v: it must be in a crossing, and since the only remaining crossings involve $C_{5,2}$ and a vertex colored 1 or 3, this forces another vertex colored 1 or 3 on the 5-2-exterior, a contradiction.

Thus we may assume that the (4,5,3,1) crossing is (mostly) on the 5-2-interior. The vertex colored 4 in the (4,5,1,2) crossing and the vertex colored 4 in the (4,5,3,2) crossing must both be in the 5-2-interior, for if not then it would force the vertices labeled 1 and 3 in these crossings to be on the 5-2-interior, but that is no longer Case 7. However, there must be a vertex colored 4 on the 5-2-exterior, on $C_{4,1}$ between the (4,5,1,2) crossing and the neighbor of v colored 1, which is a contradiction since there are only three vertices colored 4 overall. So Case 7 is impossible.

Case 8:

The vertex colored 4 in the (4,5,3,2) crossing must be on the 5-2-exterior as the 3 is on the 5-2-interior. This vertex, together with v and the two non-neighbors of v colored 1 on the 5-2-exterior, form an independent set.

Thus, in every case we find four independent vertices in G. □

Corollary 2.2. *If a graph G is drawn with exactly four crossings, which are pairwise independent, then G is 5-colorable.*

Proof. Apply Theorem 2.1 to obtain an independent set S. $G - S$ is planar so it can be 4-colored. Color S with a fifth color; now G is 5-colored. □

Now we look at the case of a graph drawn with 5 independent crossings. In the example below, there are four disjoint copies of K_5 that contain all the vertices (each containing one vertex of the middle crossing). An independent set can contain at most one vertex from each copy of K_5.

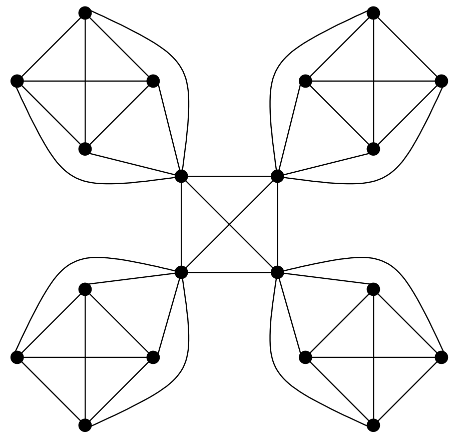

FIGURE 2. An example with 5 independent crossings, but no independent set of size 5.

References

[1] Michael O. Albertson, Chromatic Number, Independence ratio, and Crossing Number, preprint (2008)

[2] Oystein Ore, Graphs and Their Uses, The Mathematical Association of America, 1990